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ON THE EXISTENCE AND VELOCITY OF PROPAGATION OF NONLINEAR STEADY WAVES

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The problems of existence and of the upper bound of the velocity of propagation of simple steady waves for the nonlinear wave equation which arises particularly in the analysis of signal transmission in an active RCL line are investigated. It is shown that simple steady waves do exist under certain conditions which the parameters of the nonlinear medium (the line parameters) must satisfy and that the velocity of propagation of these waves does not exceed a certain value which is strictly smaller than the limiting wave propagation velocity in the medium.

The investigation of simple steady waves in nonlinear media associated either with the asymptotic transition of the system from one equilibrium state to another or with return to the initial state is of great practical importance. We need merely point to such physical phenomena as the propagation of a normal combustion front [1], excitation in a neuristor line [2], and a whole series of processes in distributed semiconductor systems such as the Gunn effect [3].

Let us consider the nonlinear wave equation

$$D\left[\frac{1}{s^2}\frac{\partial^2 c}{\partial t^2}-\frac{\partial^2 c}{\partial x^2}\right]+\left[1-\frac{D}{s^2}\frac{dQ}{dc}\right]\frac{\partial c}{\partial t}=Q(c)\qquad (D=\text{const}) \tag{1}$$

where s is the limiting wave propagation velocity and $Q(c)$ is the nonlinear "source".

As $s\to\infty$ Eq. (1) degenerates into the nonlinear diffusion equation

$$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2}+Q(c) \tag{2}$$

As noted above, wave equation (1) can be arrived at by analyzing signal transmission in an active RCL transmission line described by a system of nonlinear telegraphic equations for the form

$$-\frac{\partial\varphi}{\partial x}=Rj+L\frac{\partial j}{\partial t},\qquad -\frac{\partial j}{\partial x}=C\frac{\partial\varphi}{\partial t}+J(\varphi) \tag{3}$$

where R, C and L are, respectively, the resistance, capacitance, and inductance per unit length and $J(\varphi)$ is the nonlinear leakage current. System (3) defines the distribution of the potential $\varphi(x, t)$ and of the current density $j(x, t)$ along the line. Eliminating $j(x, t)$ from system (3), we obtain an equation analogous to (1) for the potential $\varphi(x,t)$; here $s^2 = (LC)^{-1}$, $D = (RG)^{-1}$.

Among the solutions of Eq. (1) we have a certain interesting class which corresponds to simple steady waves such that $c(x,t) = c(x + ut) = c(\eta)$. The quantity u is the wave propagation velocity. Equation (1) here becomes an ordinary first-order differential equation. Introducing the phase density pc, we obtain a first-order equation for the function $p = p(c)$,

$$\frac{dp}{dc} = \frac{u[1 - Ds^{-2}Q(c)]\,p - Q(c)}{D(1 - u^2 s^{-2})\,p} \qquad \left(p \equiv \frac{dc}{d\eta}\right) \tag{4}$$

The authors of [4] carried out a comprehensive investigation of simple steady waves for a nonlinear diffusion equation ($s \to \infty$) in the case of a function $Q(c)$ if fixed sign which vanishes for $c = 0$ and $c = 1$ and has its largest derivative at zero.

We shall deal with the case of an alternating (in sign) function $Q(c)$ which vanishes at three points ($c = 0, c_0, 1$) and a wave equation ($s \neq \infty$).

The boundary conditions are of the form

$$\lim_{\eta \to -\infty} c(\eta) = 0, \qquad \lim_{\eta \to \infty} c(\eta) = 1, \qquad \lim_{\eta \to \pm\infty} \frac{dc}{d\eta} \equiv \lim_{c \to 0;\,1} p(c) = 0 \tag{5}$$

We define a direct steady wave as a wave associated with passage from the point $c = 0$ to the point $c = 1$ for which $p \equiv dc/d\eta > 0$ for all $c \neq 0, 1$. Similarly, a reverse steady wave is associated with passage from the point $c = 1$ to the point $c = 0$ for which $p \equiv dc/d\eta < 0$ for all $c \neq 0, 1$. Equation (4) has three singular points, two of which (namely (0, 0) and (1, 0)) are singular points of the "saddle" type, while the third (namely (c_0, 0)) is a singular point of the "node" or "focus" type, depending on the sign of the inequality

$$u^2 \gtrless 4D\left(\frac{dQ}{dc}\right)_{c_0}\left[1 + \frac{D}{s^2}\left(\frac{dQ}{dc}\right)_{c_0}\right]^{-2} \tag{6}$$

A simple steady wave on the phase plane corresponds to an integral curve which extends from saddle to saddle and lies in the upper or lower half-plane for a direct or reverse wave, respectively. We note that the required trajectory is a noncoarse phase trajectory.

We make the following assumptions.

1°. The function $Q(c)$ is single-valued and continuously differentiable with a bounded derivative,

$$Q(c) < 0, \quad 0 < c < c_0, \quad Q(c) > 0, \quad c_0 < c < 1$$
$$Q(0) = Q(c_0) = Q(1) = 0, \quad Q'(0) = -\alpha,$$
$$Q'(1) = -\gamma \tag{7}$$

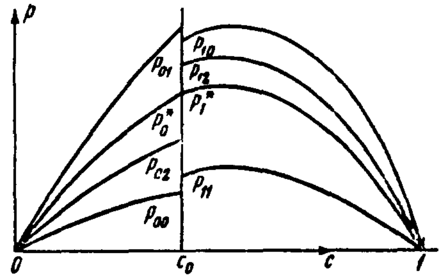

Fig. 1

2°. The parameters of the nonlinear medium are such that the inequality

$$Ds^{-2}\max_{0 \le c \le 1} Q'(c) < 1$$

$$I(1) > 0 \left(I(c) = \int_0^c Q(c)\,dc\right) \tag{8}$$

holds.

To prove the existence of a simple steady wave we need merely prove the existence of a common separatrix of the two saddle points.

Theorem 1. Let the function $Q(c)$ satisfy Conditions 1° and 2°. Then there exists a unique value of the parameter $u = u^*$ for which the positive solution of Eq. (4) connects the saddle-type singular points.

Proof. We denote the separatrices of the saddles (0, 0) and (1, 0) lying in the upper half-plane by $p_0(c)$ and $p_1(c)$ for an arbitrary value of the parameter u and by $p_{0i}(c)$ and $p_{1i}(c)$ for $u = u_i$ (Fig. 1). The angular coefficients of the separatrices are easy to determine by solving the characteristic equation for Eq. (4) linearized in the neighborhoods of the singular points.

These coefficients for the points (0, 0) and (1, 0) turn out to be

$$\begin{aligned}\lambda_0(u) &= (2D\omega)^{-1}\left[u(1+Ds^{-2}\alpha)+\sqrt{u^2(1-Ds^{-2}\alpha)^2+4D\alpha}\right]\\ \lambda_1(u) &= (2D\omega)^{-1}\left[u(1+Ds^{-2}\gamma)-\sqrt{u^2(1-Ds^{-2}\gamma)^2+4D\gamma}\right]\end{aligned} \tag{9}$$

respectively, where $\omega = 1 - u^2s^{-2}$.

It is clear that $\lambda_0(u)$ and $\lambda_1(u)$ are increasing functions of the parameter u.

Now let us consider the "degenerate" equation ($u = 0$)

$$p\frac{dp}{dc} = -\frac{1}{D}Q(c) \tag{10}$$

Integrating (10), we obtain the following equations for the separatrices of the degenerate equations

$$p_{00}^2(c) = -\frac{2}{D}\int_0^c Q(x)\,dx, \qquad p_{10}^2(c) = \frac{2}{D}\int_c^1 Q(x)\,dx \tag{11}$$

The curves $p_{00}(c)$ and $p_{10}(c)$ intersect the straight line $c = c_0$ at points determined by the function $Q(c)$.

Integrating Eq. (4), we obtain the relations which the separatrices must satisfy for arbitrary values of the parameter u,

$$p_0^2(c) = \frac{2}{D\omega}\left\{u\int_0^c [1 - Ds^{-2}Q'(x)]\,p_0(x)\,dx - \int_0^c Q(x)\,dx\right\} \tag{12}$$

$$p_1^2(c) = \frac{2}{D\omega}\left\{-u\int_c^1 [1 - Ds^{-2}Q'(x)]\,p_1(x)\,dx + \int_c^1 Q(x)\,dx\right\} \tag{13}$$

The right side of Eq. (4) is an increasing function of the parameter u for all $0 \leqslant c \leqslant 1$ and $p > 0$; hence, making use of Chaplygin's theorem on differential inequalities and of the properties of the angular coefficients of the separatrices $\lambda_0(u)$ and $\lambda_1(u)$, we can show (as is done in [5]) that for $u_1 > u_2 > 0$ the following inequalities are valid:

$$p_{01}(c) > p_{02}(c) > p_{00}(c), \quad 0 < c \leqslant c_0 \tag{14}$$

$$p_{11}(c) < p_{12}(c) < p_{10}(c), \quad c_0 \leqslant c < 1 \tag{15}$$

The disposition of these separatrices is shown in Fig. 1. In addition, we can make use of the second condition of (8) and formulas (11) to prove the validity of the inequality

$$p_{s0}(c_0) < p_{10}\ (c) \tag{16}$$

It is now easy to prove that the right side of Eq. (4) satisfies the conditions of the theorem on the continuous dependence of the solution on the parameter u. Making use of this theorem and taking into account the properties (14) and (15), we conclude that $p_0(c_0)$ and $p_1(c_0)$ are continuous monotonically increasing and monotonically decreasing (respectively) functions of the parameter u. Moreover, $p_0(c_0)$ increases without limit as $u \to s$, since (12) gives us

$$p_0^2\,(c_0) > -\frac{2}{D\omega}\int\limits_0^{c_0} Q\,(x)\,dx$$

where the ω defined in (9) tends to zero as $u \to s$. It is now clear that there exists a unique parameter value $u = u^* < s$ such that $p_0^*(c) = p_1^*(c_0)$. The theorem has been proved.

Note 1. A similar theorem can be proved for a reverse steady wave by replacing the second inequality of Condition 2° by its opposite, namely by $I(1) < 0$.

Note 2. There is no reverse wave if $I(1) > 0$. This can be demonstrated by reductio ad absurdum: the opposite statement and the properties of $Q(c)$ imply the existence of a point $c_0 < c^* < 1$ such that $I(c^*) = 0$; this result and relation (12) for $c = c^*$ imply the contradiction of the premise. Thus, $u = u^*$ is the velocity of propagation of simple steady waves. It turns out, moreover, that u^* is smaller than a quantity which is strictly smaller than the limiting propagation velocity s. The following theorem gives us an upper estimate for the velocity of propagation of simple steady waves.

Theorem 2. Let the function $Q(c)$ satisfy Conditions 1° and the first inequality of (8). The following estimates for the velocity of propagation of a simple steady wave then hold:

$$u^* < u^{\pm} < s, \qquad u^{\pm} \equiv \frac{2\sqrt{DQ^{\pm}}}{1 + Ds^{-2}Q^{\pm}} \tag{17}$$

where the plus and minus superscripts refer to the direct and reverse waves, respectively, and

$$Q^+ = \max_{c_0 \leqslant c \leqslant 1} Q'(c), \quad Q^- = \max_{0 \leqslant c \leqslant c_0} Q'(c) \tag{18}$$

Proof. The following proof is based on a simple generalization of the device employed in [4].

Let us suppose that the theorem is untrue (i.e. that values $u > u^+$ are permissible) and prove that simple steady waves do not exist. Let us consider the isoclines of Eq. (4),

$$k \equiv \left(\frac{dp}{dc}\right)_{(c,\,p)} = \frac{u\,[1 - Ds^{-2}Q'(c)]\,p - Q\,(c)}{D\omega p}$$

The equation of the family of isoclines has the form

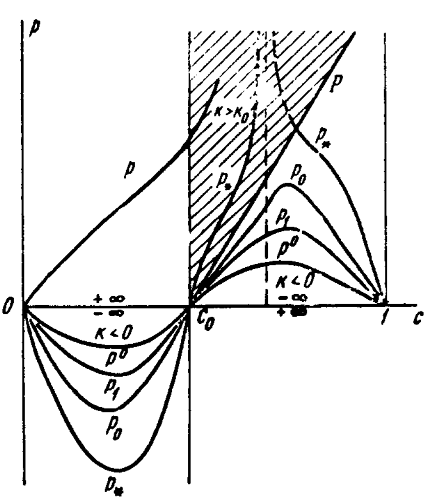

Fig. 2

$$p_k(c) = \frac{Q(c)}{u[1 - Ds^{-2}Q'(c)] - kD\omega} \tag{19}$$

For $0 \leqslant k < k_* = u\Delta / (D\omega)$, where $\Delta = 1 - Ds^{-2}Q^+$, the curves $p_k(c)$ are continuous and vanish (Fig. 2) at the points $c = 0,\ c_0,\ 1$ ($p^0(c)$ corresponds to the value $k = 0$). For $k = k^*$ the isocline $p_*(c)$ has a discontinuity at the maximum of $Q'(c)$ (we assume that $Q^+ > Q^-$; the latter fact is immaterial, however, since we are considering the behavior of the curves $p_k(c)$ for $c_0 \leqslant c \leqslant 1$ only).

It is easy to show that for $c_0 < c < 1$ and any $k_1 < k_2 < k^*$ we have $p_1(c) < p_2(c)$, where the subscripts correspond to the values k_1 and k_2.

Let us choose the isocline corresponding to the value

$$k_0 = (2D\omega)^{-1}[u\Delta + \sqrt{u^2\Delta^2 - 4D\omega}] < k^* \tag{20}$$

Here $\Delta = 1 - Ds^{-2}Q^+$, and $\omega = 1 - u^2s^{-2}$. It is clear that the parameter k_0 is real by virtue of our assumption. The equation of the isocline is

$$p_0(c) = \frac{2Q(c)}{u\Delta + 2Dus^{-2}[Q^+ - Q'(c)] - \sqrt{u^2\Delta^2 - 4D\omega Q^+}} \tag{21}$$

By virtue of the foregoing considerations, all of the isoclines corresponding to $0 < k < k_0$ lie between the zeroth isocline and isocline (21). Let us construct the straight line $P(c) = k_0(c - c_0)$. The latter has only one point of intersection with the isocline $p_0(c)$ in the range $c_0 \leqslant c \leqslant 1$, namely the point $(c_0, 0)$. If this was not so, the equation $p_0(c) = P(c)$ would have a root for $c \neq c_0$. But this is impossible since the equation under consideration is equivalent to the equation

$$Q(c) = (c - c_0)\ [Q^+ + u^2k_0Ds^{-2}(Q^+ - Q'(c))]$$

which has only one root $c = c_0$ with Q^+ chosen as above.

Thus, the angular coefficients of the integral curve are larger than k_0 for all points of the phase plane lying in the half-strip $c_0 < c < 1$ above and on the straight line $P(c) = k_0(c - c_0)$. (This is the shaded area in Fig. 2).

Hence, an integral curve $p(c)$ which intersects the straight line $c = c_0$ above the axis does not intersect the straight line $P(c)$; this excludes the possibility of an integral curve $p(c)$ passing through the point (1, 0). Finally, we conclude that an integral curve lying in the upper half-plane and connecting the points (0, 0) and (1, 0) does not exist. Hence, inequalities (15) for the velocity of propagation of a simple steady wave are valid.

The theorem has been proved. The proof for the reverse wave can be carried out in precisely similar fashion.

Note 1. The first condition of (8) isolates the domain of values of the basic parameters of the nonlinear media corresponding to the propagation of "slow" waves (waves of velocity smaller than the limiting velocity). For an active RCL transmission line this condition becomes

$$RC > L \max J'(\varphi) \qquad (0 \leqslant \varphi \leqslant 1)$$

Thus, if the line parameters R, C and L are such that the above condition is satisfied, then the existence of simple steady waves propagating at velocities strictly smaller than the limiting velocity $s = 1 / \sqrt{LC}$ is guaranteed. The possibility of regulating the velocity of signal transmission in an active line is of great practical interest.

If the opposite inequality holds in the first condition of (8), then (as was verified by numerical calculations) continuous simple steady waves can propagate at velocities close to the limiting velocity s in the case of a piecewise-linear approximation of the function $Q(c)$; the steepness of the front of such a wave increases without limit, and discontinuous waves can propagate in the medium for $u = s$. Moreover, the indicated condition has an important bearing on the uniqueness of the solution of problem (4), (5).

Note 2. The quantities $u^{\pm}$ coincide with each other and with the estimate given in [6] if the maximum value of the derivative $Q'(c)$ is realized at the point $c = c_0$ and if the function is strictly convex and strictly concave in the ranges $0 \leqslant c \leqslant c_0$ and $c_0 \leqslant c \leqslant 1$, respectively.

Note 3. The first condition of (8) is not essential in the "diffusion" approximation $(s \to \infty)$, and all the results are valid under weaker restrictions on $Q(c)$. Moreover, the inequalities can be refined by replacing $Q^{\pm}$ by the minimum values of the angular coefficients of the curves $k^{\pm}(c - c_0)$, which do not intersect $Q(c)$ for $c_0 \leqslant c \leqslant 1$ and $0 \leqslant c \leqslant c_0$, respectively, except at the point $(c = c_0, p = 0)$.

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